

Conjectured \mathbb{Z}_2 -Orbifold Constructions of Self-Dual Conformal Field Theories at Central Charge 24 - the Neighborhood Graph

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February 1, 2008

Abstract

By considering constraints on the dimensions of the Lie algebra corresponding to the weight one states of \mathbb{Z}_2 and \mathbb{Z}_3 orbifold models arising from imposing the appropriate modular properties on the graded characters of the automorphisms on the underlying conformal field theory, we propose a set of constructions of all but one of the 71 self-dual meromorphic bosonic conformal field theories at central charge 24. In the \mathbb{Z}_2 case, this leads to an extension of the neighborhood graph of the even self-dual lattices in 24 dimensions to conformal field theories, and we demonstrate that the graph becomes disconnected.

1 Introduction

The problem of the classification of two-dimensional conformal field theories has seen much activity and significant progress over the last decade. While many approaches rely on, for example, the classification of fusion rules of some chiral algebra [32], such methods ignore those theories for which these fusion rules are trivial. These theories however are themselves far from trivial, an example being the natural module for the Monster group, the so-called Moonshine module [11]. Thus, these “self-dual” theories must be classified separately from this mainstream approach.

While such theories are of interest in their own right and serve as a simpler arena in which to understand the general structure of conformal field theory, they also find physical application. For example, the classification of the self-dual theories at central charge 24 is relevant to the classification of 10 dimensional heterotic strings [25]. There is a partial classification of these theories due to Schellekens [27, 25, 26]. Of the 71 potential theories (under assumptions we will discuss below), explicit constructions have only been proposed for some 43 of them [8, 27, 20]. In this paper, we shall propose orbifold constructions for all but one of the remaining theories by using results from [20] and exploiting the modular transformation properties of the graded characters of automorphisms of orders two and three to obtain information regarding the Lie algebraic content of any proposed orbifold model.

There is also considerable physical motivation for studying orbifold models in general. Many of the interesting applications of conformal field theory involve twisted fields, for example in the analysis of critical phenomena, but particularly in the construction of physically realistic (super)string models [3, 4, 5].

We shall also show that our main result is a generalisation of a corresponding result in the case of even self-dual lattices in 24 dimensions. Such analogies between the theory of lattices and conformal field theories, as described in [14, 8, 19] and illustrated by [25] (see our comments below), are fruitful in that they enable one to make conjectures based on our understanding of the structures at simpler levels. It is to be hoped that such investigations will shed light on the intricate structures involved in conformal field theory.

2 Modular constraints on dimensions

Let \mathcal{H} be a meromorphic chiral bosonic conformal field theory of central charge c [8] [note that from now on by the term conformal field theory we will understand this more restricted structure]. The modes of the vertex operators of the states of conformal weight one in \mathcal{H} form an affine Lie algebra $\widehat{\mathfrak{g}}_{\mathcal{H}}$ [14]. Further, we say that \mathcal{H} is self-dual if its “partition function”

$$\chi_{\mathcal{H}}(\tau) \equiv \text{Tr}_{\mathcal{H}} q^{L_0 - c/24}, \quad (1)$$

where $q = e^{2\pi i\tau}$, is a modular invariant function of τ . [This is necessary for the chiral theory to be well-defined on a torus, see *e.g.* [13]. For more discussion of possible distinct definitions of self-duality see [21].] The self-dual conformal field theories can only exist at central charges a multiple of 8. At central charges 8 and 16, the theories are easily classified [14]. They are all given by the Frenkel-Kac-Segal [FKS] construction [12, 28] from an even self-dual lattice Λ of dimension d equal to the corresponding central charge. [Physically this describes the propagation of a bosonic string on the torus \mathbb{R}^d/Λ .] We denote such a conformal field theory by $\mathcal{H}(\Lambda)$. In 32 dimensions, the number of even self-dual lattices alone [2] is such that an explicit enumeration is infeasible. Therefore central charge 24 is the last possible case which may be amenable to classification. In 24 dimensions there are 24 even self-dual lattices [31], and in [8, 7] it was shown that the corresponding theories $\mathcal{H}(\Lambda)$ together with their reflection-twisted orbifolds $\widetilde{\mathcal{H}}(\Lambda)$ (see below) give 39 inequivalent self-dual conformal field theories of central charge 24. In [25, 26] using a beautiful result partially analogous to Venkov's classification of the even self-dual lattices in 24 dimensions [31], Schellekens identified all possible affine algebras which can correspond to a self-dual conformal field theory at central charge 24. There are 71 such algebras (each with distinct Lie algebra), and in each case he identified a unique modular invariant combination of affine algebra characters. The existence and uniqueness of a conformal field theory corresponding to each algebra remains to be established. Throughout this paper we shall assume uniqueness (there are certainly no known counter-examples) and address the problem of existence. The main result of this paper is to suggest possible constructions of most of the 71 theories and to further reinforce the analogies with the theory of lattices emphasized in [19].

We now recall the results of [20]. Let \mathcal{H} be a self-dual conformal field theory at central charge 24. The partition function for such a theory is simply of the form $J(\tau) + c$, where c is the number of states of conformal weight one and J is the classical modular function (with zero constant term). Consider an automorphism θ of \mathcal{H} of order two. We suppose that the orbifold of \mathcal{H} with respect to θ exists and is a consistent conformal field theory (for example for the involution induced by the reflection on the lattice, it was shown in [6] that the corresponding orbifold $\widetilde{\mathcal{H}}(\Lambda)$ of the (self-dual) Frenkel-Kac-Segal conformal field theory $\mathcal{H}(\Lambda)$ constructed from an even self-dual lattice Λ is consistent). In other words, if we let \mathcal{H}_0 denote the sub-conformal field theory [8] invariant under θ , then there exists a "twisted representation" \mathcal{K}_0 of \mathcal{H}_0 such that $\widetilde{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{K}_0$ is a consistent conformal field theory (with an obvious definition of the vertex operator structure). We assume that this orbifold conformal field theory is also self-dual. (Under rather natural physical assumptions, it is trivial to see that the partition function is modular invariant - see *e.g.* [20].) We write

its partition function as $J(\tau) + c'$. In [20] it was shown, again under certain natural physical assumptions on the modular transformation properties which have recently been shown [9] to be equivalent to the statement that the ground state of the twisted representation \mathcal{K} of \mathcal{H} corresponding to the representation \mathcal{K}_0 of \mathcal{H}_0 has conformal weight in $\mathbb{Z}/2$, *c.f.* Vafa's "level matching" condition [30], that

$$c + c' = 3c_0 + 24 - 24\alpha, \quad (2)$$

where c_0 is the number of states of conformal weight one in \mathcal{H}_0 and α is the number of states of conformal weight $1/2$ in \mathcal{K}' .

Since the zero modes of the states of conformal weight one give the Lie algebra g_V of a conformal field theory V , this is simply a statement about the relation between the dimensions of the Lie algebras corresponding to \mathcal{H} , $\tilde{\mathcal{H}}$ and \mathcal{H}_0 .

There is no known general procedure for writing down the twisted sector corresponding to a given automorphism θ of an arbitrary conformal field theory (except in the case of theories of the form $\mathcal{H}(\Lambda)$ – see *e.g.* [18, 15], but note the missing terms arising from normal ordering, as discussed in [6, 22]). However, we may use the above result to obtain a set of possible values for the dimension of the Lie algebra of such an orbifold. It is given by

$$3c_0 + 24 - c - 24\alpha, \quad (3)$$

for some non-negative integer α , and must be at least c_0 .

3 Main method

Now, to a given \mathbb{Z}_2 -orbifold construction, there exists an inverse \mathbb{Z}_2 -orbifold construction with involution defined to be $+1$ on \mathcal{H}_0 and -1 on \mathcal{K}_0 [29, 20]. Since an automorphism of a conformal field theory preserves the conformal weights [8], then it restricts to an automorphism of the corresponding Lie algebras (of order dividing the order of the original automorphism). In other words, the Lie algebra of the invariant theory \mathcal{H}_0 should be either equal to or a \mathbb{Z}_2 -invariant subalgebra of *both* $g_{\mathcal{H}}$ and $g_{\tilde{\mathcal{H}}}$.

The method then is to consider in turn in order of decreasing dimension each of the algebras on Schellekens' list of 71 algebras at central charge 24 for which we already have a constructed theory, and for each to consider all possible \mathbb{Z}_2 invariant subalgebras g of the Lie algebra. We evaluate the possible dimensions of the algebra of the orbifold theory by the above, and then check to see whether any of the theories on Schellekens' list at the corresponding dimension has g as a \mathbb{Z}_2 invariant subalgebra. Note that by making the trivial observation of the existence of the orbifold inverse, we have avoided the need to consider the \mathbb{Z}_2 invariant subalgebras of a given Schellekens theory as sitting as an arbitrary subalgebra

inside some other Schellekens Lie algebra. Such calculations would be complicated by the need to consider so-called exceptional subalgebras (see, for example, [10]). It is interesting to note that in the following set of results, that there would exist only a few spurious solutions in which the embedding is not a \mathbb{Z}_2 invariant. We regard this as a testament to the power of our method.

In order to reduce the size of the calculation to a more manageable form (and avoid resort to computer calculation) we ignore potential constructions of theories which we already have constructed, *i.e.* initially just the 39 theories $\mathcal{H}(\Lambda)$ and $\widetilde{\mathcal{H}}(\Lambda)$ for the 24 even self-dual lattices Λ . For example, we have all of the theories of rank 24 (in [8] it was shown that if $\text{rank}_{\mathcal{H}} = c$ then $\mathcal{H} \cong \mathcal{H}(\Lambda)$ for some even lattice Λ , self-dual if \mathcal{H} is self-dual), and hence we need consider only those (outer) automorphisms of $g_{\mathcal{H}(\Lambda)}$ which reduce the rank to less than or equal to 16 (the next rank below 24 on Schellekens' list).

Once we identify a given \mathbb{Z}_2 automorphism of a given theory as giving potentially a unique new theory, we then expand our considerations for this automorphism to include all theories at the appropriate dimensions, *i.e.* include those that we have already constructed. If the new theory is still a unique candidate, then we regard this as a construction, note it in Table 1, and add that theory to our list of constructed theories.

Thus we obtain a list of conjectured constructions of theories on Schellekens' list. Note though that to verify each construction, we must extend the automorphism from the Lie algebra of the initial theory to the whole conformal field theory (we make some comment on this below in specific cases) and then verify that the orbifold theory is consistent (along the lines of [6]) by writing down a set of vertex operators for a twisted sector and verifying the appropriate locality relations hold true. This is a difficult problem, and is left to future work.

Perhaps though the real power of this method is in showing which constructions *cannot* produce consistent theories and hence restricting effort to those which are potentially fruitful. As a simple example, one might consider that the common \mathbb{Z}_2 invariant subalgebra $A_4 A_3 U(1)$ of $A_4^2 C_4$ and A_4^2 might indicate a possible orbifolding of one from the other. However, the dimension of the Lie algebra of the orbifold theory of $A_4^2 C_4$ corresponding to an automorphism with invariant algebra $A_4 A_3 U(1)$ is, by our method, $60 - 24\alpha$ for some non-negative integer α , whereas the dimension of the theory A_4^2 is 48. Thus the orbifold theory cannot be consistent (or perhaps the automorphism does not even lift from the Lie algebra to the conformal field theory). In other words, the additional input obtained from considerations of modularity properties helps to eliminate such spurious and naive solutions. One would of course be tempted to hypothesize that all such solutions which satisfy the modularity constraint correspond to consistent orbifolds, though we will find a counterexample

later when we come to consider the theory A_2^{12} , at least in so far as the automorphism may still fail to extend to the full conformal field theory.

Once we have considered all of the 39 theories $\mathcal{H}(\Lambda)$ and $\widetilde{\mathcal{H}}(\Lambda)$, we may then consider orbifolding the orbifolds which we have constructed so far. In fact, this can easily be seen to be a necessary procedure, since the rank of the orbifolded theory (with the exception of orbifolds of the algebra $U(1)^{24}$ – see later for a full analysis of this case) is clearly at least half of that of the original theory. Since there is one theory on Schellekens' list of rank 4, then we see that at least 3 successive orbifoldings from the set of theories $\mathcal{H}(\Lambda)$ need be considered.

Having considered many of the theories in this way, towards the end we switch techniques and look for orbifolds of the as yet unconstructed theories in the hope of linking them by a \mathbb{Z}_2 -orbifold either to each other or to an already constructed theory. This is obviously a more efficient technique when only a few theories remain to be found.

4 Results and Comments

Obviously we will not give full details of the calculations, since they are mainly a case by case trivial application of the above elementary arguments. We give a selection of examples only for illustrative purposes, and refer the reader to Table 1 for a more detailed summary of the results.

We first recall from [17] the following theorem.

Theorem Let g be a simple Lie algebra and let the (extended) Dynkin diagram $D(g^\tau)$ of $g^{(\tau)}$, $\tau = 1, 2, 3$, have Kac labels k_i^τ , $1 \leq i \leq O(g, \tau)$. Let $s_0, \dots, s_{O(g, \tau)}$ be a sequence of non-negative relatively prime integers and set $N = \tau \sum_{i=0}^{O(g, \tau)} k_i^\tau s_i$. Then the conjugacy classes of the automorphisms of order N of g are in one-to-one correspondence with the sequences s_i which cannot be transformed into one another by an automorphism of $D(g^\tau)$. Further, the invariant Lie subalgebra is isomorphic to the direct sum of the semi-simple Lie algebra obtained by removing all the vertices from $D(g^\tau)$ corresponding to non-zero s_i together with a centre $U(1)^{O(g, \tau) - r}$, where r of the s_i 's vanish.

Consider first the theory (with Lie algebra) D_{24} . [By our previous comment regarding the rank 24 cases, this (is unique and) must be an FKS theory.] From the $D_{24}^{(1)}$ and $D_{24}^{(2)}$ Dynkin diagrams together with the above theorem, we see that the rank of the invariant Lie subalgebra (and hence of any orbifold theory) is at least 23, and Schellekens' list then tells us that the rank of the orbifold must be 24 if the theory is to be consistent. All such theories, as discussed above, are already constructed. Hence we have no calculation to do in

this case – all \mathbb{Z}_2 orbifolds can only give theories of the form $\mathcal{H}(\Lambda)$ for Λ even and self-dual. For example, the reflection twist [6] gives us the theory corresponding to the Niemeier lattice D_{12}^2 [8].

Consider now the theory $A_{17}E_7$. Again this is simply an FKS theory. Considering the appropriate Dynkin diagrams, we see that possible \mathbb{Z}_2 invariant subalgebras of E_7 are $E_6U(1)$, A_1D_6 and A_7 . We must also consider E_7 itself as the algebra corresponding to a \mathbb{Z}_2 invariant sub-conformal field theory in which the automorphism is trivial on the states of weight one. For the A_{17} factor, we need only consider outer automorphisms, since otherwise the rank of the invariant algebra will be 24 and we will not obtain a new theory. The possible algebras that we get from outer automorphisms are D_9 and C_9 . There are thus eight possible invariant subalgebras, all of rank 16. At this point it is worth making the rather trivial observation that all theories on Schellekens list have dimensions which are a multiple of 12. In order that the dimension from (3) be a multiple of 12, we require the dimension of the invariant subalgebra to be a multiple of 4. This immediately excludes half of our eight possible algebras, leaving

1. $D_9E_6U(1)$ The dimension of the “enhanced” (*i.e.* orbifold) Lie algebra is, from (3), $264 - 24\alpha$ and must be at least the dimension of the subalgebra, *i.e.* 232. Thus we must consider theories with dimensions 240 or 264. At 264, there are no new theories (*i.e.* theories of which we do not already have a construction). At 240, there are new theories, but none which admit the invariant subalgebra as a subalgebra (\mathbb{Z}_2 -invariant or otherwise) We conclude that this automorphism (or more precisely the class of automorphisms with invariant algebra isomorphic to this) does not produce any new theories by orbifolding.
2. D_9A_7 In this case, the upper bound on the dimension of the algebra of the orbifold model from (3) and the lower bound (*i.e.* the dimension of the \mathbb{Z}_2 -invariant algebra) coincide. Thus, if the theory is to be consistent, it must have algebra isomorphic to the invariant algebra. Such a theory exists on Schellekens’ list. In fact, this case simply corresponds to the automorphism induced by the reflection on the lattice, *i.e.* the orbifold theory is $\widetilde{\mathcal{H}}(\Lambda)$, and is known to be consistent [8]. See the comments below regarding the special status of the reflection twisted orbifolds.
3. C_9E_7 As in case 1, we conclude that this class of automorphisms cannot give rise to any new theories.
4. $C_9A_1D_6$ The “new” theories of dimension 288, 264 or 240 are B_6C_{10} , $B_5E_7F_4$ and $C_8F_4^2$. Only the former has $C_9A_1D_6$ as a subalgebra (\mathbb{Z}_2 -invariant or otherwise). Hence this is a possible candidate for a new theory. To assign a greater degree of confidence to this construction, we expand our considerations to all the theories on Schellekens’

list at the above range of dimensions. Having done this, we observe that the theory B_6C_{10} remains the unique candidate. We conjecture that this orbifold construction is consistent and produces this theory.

It is worthwhile at this point to consider the nature of the relation between this consistency test and Vafa’s “level-matching” condition [30], *i.e.* that the conformal weight of the twisted sector ground state is a half-integer. [In fact not only should we check in which cases the possible consistency of an orbifold theory is agreed upon, but we may also check that cases in which the conformal weight of the ground state in the twisted sector is at least 1 correspond to the cases $\alpha = 0$, *i.e.* no states at conformal weight $1/2$ in the twisted sector.] As mentioned earlier, it has been shown [9] that the level-matching condition in the \mathbb{Z}_2 case is equivalent to the assumption that the character of the automorphism transforms appropriately under modular transformations (*i.e.* that it is a $\Gamma_0(2)$ invariant). However, our test is not *equivalent* to level-matching, since we also use non-trivial input by checking potential theories against Schellekens’ list. For example, considering the invariant algebra $E_8^2 E_7 A_1$ of E_8^3 , the conformal weight of the twisted sector ground state should, by standard results [24], be $1/4$ and so the orbifold should be inconsistent. However, our test here cannot exclude the possibility that the orbifold theory is E_8^3 again. On the other hand, take the invariant algebra $A_5^4 A_1^4$ of E_6^4 . According to the considerations here, we find that it cannot give a consistent orbifold theory. However, from [24] we see that the twisted sector ground state should have conformal weight 1.

In general though, we conjecture that a \mathbb{Z}_2 orbifold of a self-dual theory is consistent if and only if the ground state of the twisted sector has conformal weight a half-integer. The apparent counterexample in the case of E_6^4 immediately above we attribute to the fact that the automorphism under consideration does not lift to the conformal field theory as an involution (it is easy to see that any inner automorphism of the Lie algebra of a FKS theory lifts to the whole theory (for an outer automorphism one may encounter problems with the glue vectors [2]), but the appropriate definition on the cocycles [6] may lead to a change in the order of the automorphism in its action on the full conformal field theory). Note that a restriction such as the phrase “self-dual” is necessary, since in the case of a reflection-twist $\widetilde{\mathcal{H}}(\Lambda)$ of an FKS theory $\mathcal{H}(\Lambda)$ the orbifold is consistent if and only if $\sqrt{2}\Lambda^*$ is an even lattice [19], but the conformal weight of the twisted sector ground state is a half-integer for all even lattices Λ .

Our proposed test therefore, while perhaps not strictly stronger than simple level-matching, can obviate the need to consider the extension or otherwise of automorphisms to the full conformal field theory, and in cases as we will consider below in which the calculation of the conformal weight of the twisted sector ground state is impossible, it is the only indicator

for consistency of the orbifold which we have available. We rely on it in such cases, since in those other cases for which we can check level-matching (*e.g.* the FKS theories) we find very few orbifold theories which fail to be consistent on the grounds of “level-matching” and yet appear consistent to our checks. Almost all examples, as the E_8^3 case above, involve potential orbifolding back to the original theory (in which case our test is weakened since the invariant subalgebra is then automatically a \mathbb{Z}_2 -invariant subalgebra of the orbifold algebra), and so are irrelevant in any case in our attempted construction of the entire list of self-dual theories at central charge 24.

Having said that, we consider below an example which patently fails “level-matching” and yet cannot be excluded by our test. It is an example of the potential failure due to spurious solutions at low dimension which are possible. We can exclude it either by level-matching, or, as we discuss below, by showing that the automorphism does not extend to the conformal field theory. In general, one must therefore be aware that a range of techniques need be employed in order to eliminate apparently possible orbifold theories. Ultimately, all such proposed constructions must be demonstrated by explicit formulation of the appropriate vertex operators and calculation of locality relations. The current work, as already mentioned, is in many ways simply filtering out from the mass of possible constructions a handful of cases which will be likely to be consistent. At the very least, we are able to rigorously discard without excessive calculation cases which cannot be consistent.

Consider the involution of the Lie algebra A_2^{12} given by six interchanges, with invariant subalgebra A_2^6 . This cannot give rise to a consistent orbifold by level-matching, since the conformal weight of the twisted sector ground state should be $3/4$. But according to our method, the theory A_2^6 with $\alpha = 1$ is a possible (in fact the only) candidate. There is immediate evidence however that this is an accidental solution. The “enhanced” algebra (that of the orbifold theory) is equal to the invariant algebra, and so there are no states of conformal weight one in the putative twisted sector. However, $\alpha = 1$ indicates that there is a single state at conformal weight $1/2$. These facts are difficult to reconcile. It might be thought that explicit calculation of α (possible in these FKS theories) could eliminate this theory. But in fact we find we cannot even extend the automorphism from the Lie algebra up to the full conformal field theory. The glue code for the Niemeier lattice A_2^{12} is the ternary Golay code. We are required to find six disjoint transpositions in this code which leave it invariant (or equivalently, using self-duality of the code, map each basis codeword to a codeword in the dual). No such set of transpositions exists, and so the automorphism fails to extend to the conformal field theory.

Since the construction of $A_2^4 D_4$ from A_2^{12} (see Table 1) is also potentially a spurious low dimension solution, we can explicitly check in that case that the automorphism does

orbifold algebra	original algebra	invariant algebra	orbifold algebra	original algebra	invariant algebra
$E_8 B_8$	E_8^3	$E_8 D_8$	$A_2^2 A_5^2 C_2$	$A_5^4 D_4$	$A_1 A_2^2 A_3 A_5 C_2 U(1)$
$B_6 C_{10}$	$A_{17} E_7$	$A_1 C_9 D_6$	$A_4^2 C_4$	A_4^6	$A_4^2 C_2^2$
$B_5 E_7 F_4$	$D_{10} E_7^2$	$B_4 B_5 E_7$	$A_2^4 D_4$	A_2^{12}	$A_2^4 A_1^4$
$C_8 F_4^2$	$A_{15} D_9$	$B_4^2 C_8$	$A_3 C_7$	$B_4 C_6^2$	$A_1 A_3 C_6$
$B_4 C_6^2$	$A_{11} D_7 E_6$	$B_2 B_4 C_4 C_6$	$A_1 A_3^3$	$A_2^2 A_5^2 C_2$	$A_1^3 A_3^2 U(1)$
$A_5 C_5 E_6$	E_6^4	$A_1 A_5 C_4 E_6$	$A_2 C_2 E_6$	$A_3^2 D_5^2$	$A_2 C_2 D_5 U(1)$
$A_4 A_9 B_3$	A_8^3	$A_3 A_4 A_8 U(1)$	$A_1^2 C_3 D_5$	$C_2^4 D_4^2$	$A_1^3 C_2 D_4 U(1)$
$A_8 F_4$	A_8^3	$A_8 B_4$	$A_1^3 A_7$	$A_2^2 A_5^2 C_2$	$A_1^4 A_5 U(1)$
$A_3 A_7 C_3^2$	$A_7^2 D_5^2$	$A_1^2 A_3 A_7 B_2^2$	$A_2^2 A_8$	$A_4^2 C_4$	$A_1 A_2 A_3 A_4 U(1)^2$

Table 1: Conjectured orbifold constructions of 18 new theories

extend to the whole theory. The automorphism which reduces A_2 to A_1 maps the glue code 1 to 2 (in the notation of [2]). We find an appropriate symmetry of the ternary Golay code. Note though that the definition of the action of the automorphism on the cocycles could potentially lead to a doubling in the order of the automorphism. Nevertheless, we feel that the checks performed are sufficient to accept this orbifold construction as valid.

We then proceed to consider \mathbb{Z}_2 orbifolds of the theories constructed so far (*i.e.* those from the FKS theories in Table 1 together with the 15 of the theories $\widetilde{\mathcal{H}}(\Lambda)$ distinct from the FKS theories). As mentioned above, calculation of the twisted sector (*i.e.* of α and verification of level-matching) is not possible in these cases, since too little is known about the structure of the original theories. But we rely on our technique given its proven reliability in cases when cross-checks may be made. The results are as indicated in Table 1.

We conclude this section with some trivial observations.

Firstly, we note that, in order that the difference between our upper and lower bounds on the dimension of the orbifold Lie algebra be non-negative, we require that the dimension of the invariant Lie algebra, c_0 , be at least $\frac{1}{2}(c - 24)$. This value is attained for the reflection twist on the lattice. We may ask whether the converse is true, *i.e.* whether this is the only involution for which our bounds coincide. Checking all possible Lie algebra involutions, we find that for any simple Lie algebra the dimension of the invariant subalgebra is always at

least $\frac{1}{2}(d - r)$, where d is the dimension of the Lie algebra and r is its rank, and further we have equality for a unique involution in each case. Since the Lie algebra for each of the FKS theories $\mathcal{H}(\Lambda)$ for Λ Niemeier is of rank 24, then we see that we have equality of our upper and lower bounds if and only if the involution is simply the lift of the reflection twist. Thus, the reflection-twisted orbifolds $\widetilde{\mathcal{H}}(\Lambda)$ of the Niemeier lattices Λ are in some sense the extreme cases which saturate the bounds obtained from modularity constraints.

Secondly, we note that all of the constructions which we have proposed (and which we will propose in the section below) correspond to $\alpha = 0$, *i.e.* to the ground state in the twisted sector having conformal weight at least 1. We know of no reason why this should be so, since it is not necessarily always true as the simple example of the orbifolding of the E_8^3 theory with an involution which is simply the reflection twist on one E_8 factor and the identity on the other two. This gives us back the theory E_8^3 , and corresponds to $\alpha = 16$. We do not believe that it is an artifact of our method of search [unlike the fact that most of the constructions seem to relate a given theory to an orbifold theory with algebra of strictly smaller rank – this is merely a result of us beginning with the rank 24 theories and working downwards, and indeed because of the existence of an inverse to any given orbifold construction rank reduction and rank enhancement should obviously be equally common] and so remains to be explained.

In fact, for $\alpha = 0$ or 1, the graded trace of the involution is a hauptmodul. It is an interesting aside, in relation to the Moonshine conjectures of Conway and Norton [1], to note that our method shows (in all but one case – which cannot be decided unambiguously) that the “triality involution” [8, 11] (one of the few non-trivial cases of an involution known rigorously to be well-defined on a non-FKS conformal field theory) corresponds to a hauptmodul, assuming the relevant orbifold to be consistent.

5 The Neighborhood Graph

At this stage, 14 of the 71 Schellekens theories still remain to be constructed. We now, as discussed earlier, and using the fact that to each orbifold there corresponds an inverse orbifold construction, consider orbifolding in turn each of these theories and compare, using our techniques, against the *full* list of Schellekens’ theories constructed so far. We draw a graph (Figure 1) with solid arrows indicating the construction is the unique possibility and dashed arrows indicating that there is an ambiguity. Note that an ambiguity may often be resolved by considering orbifolding the theory at the opposing end of the corresponding dashed arrow. We have removed all such ambiguities (and are implicitly assuming that the class of automorphisms with a given invariant algebra produce isomorphic orbifold theories).

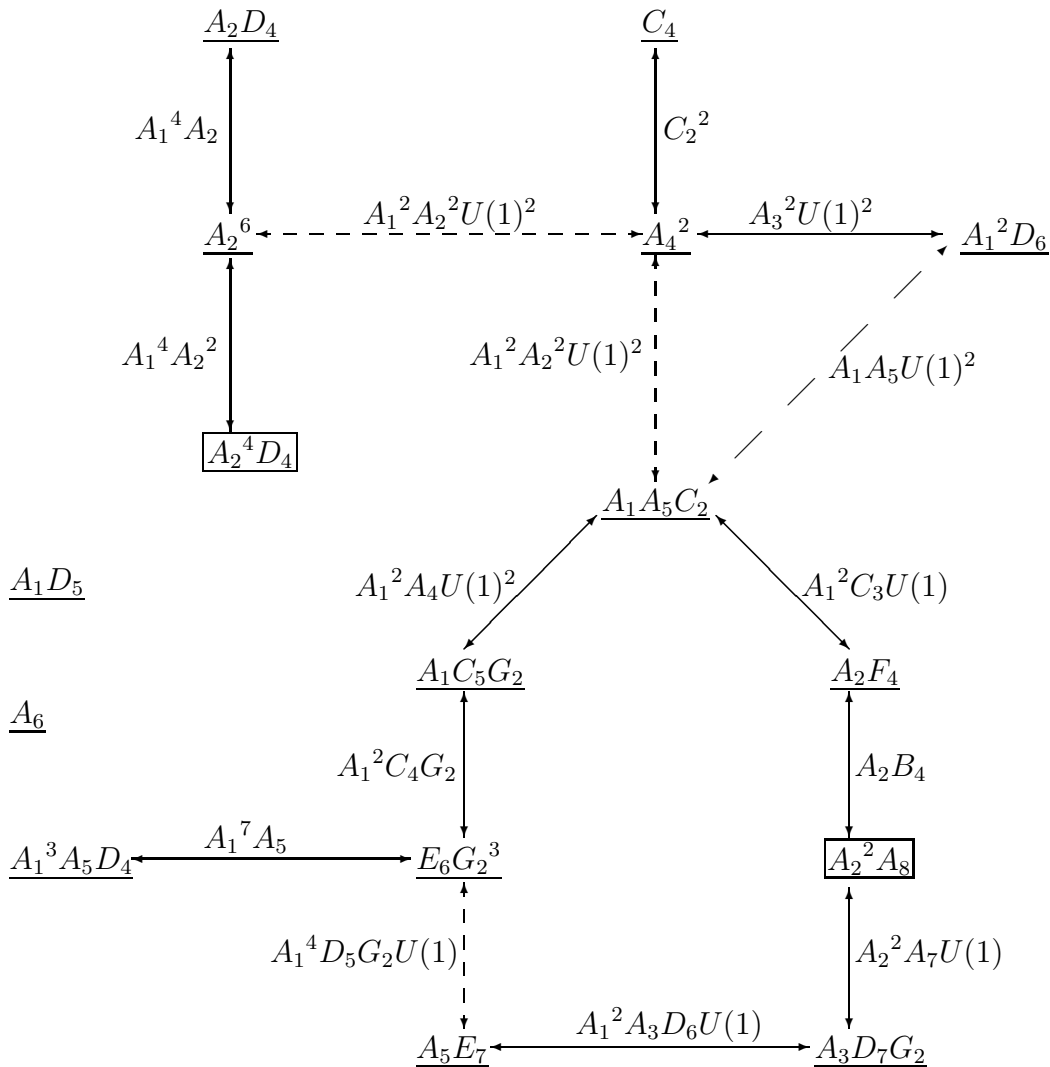


Figure 1: Subgraph of the “neighborhood graph” of the central charge 24 self-dual conformal field theories

Theories which have already been constructed in the above are enclosed in a box. We have not included all arrows coming out of a theory when there are sufficient arrows present to connect that theory (indirectly) to a theory already constructed. The appropriate invariant algebras are indicated on the connecting arrows.

We thus see that two of the theories, namely those with algebras A_1D_5 and A_6 , cannot be obtained by \mathbb{Z}_2 orbifolding, while for another three theories, C_4 , A_4^2 and $A_1^2D_6$, there is some ambiguity as to whether they may be obtained. Note that this isolation of the theories A_1D_5 and A_6 is rigorous (whereas our conjectured constructions of course remain to be verified in detail).

We now relate this picture to the “neighborhood graph” for even self-dual lattices as described by Borchers in [2]. We shall show that the complete graph of all 71 self-dual central charge 24 conformal field theories with connections corresponding to \mathbb{Z}_2 -orbifold

constructions of one theory from another has the neighborhood graph of the 24 dimensional even self-dual lattices as a sub-graph (identifying the node for the lattice Λ with that for the conformal field theory $\mathcal{H}(\Lambda)$, both of which in any case we label by the (identical) corresponding Lie algebra). The main results of this paper can be interpreted as simply extending this neighborhood graph. In 8 and 16 dimensions, the neighborhood graphs of both the lattices and conformal field theories coincide. However, this is obviously no longer the case in 24 dimensions, and in particular, in the light of the above comments regarding the theories with algebras A_6 and A_1D_5 , we see that the neighborhood graph of the central charge 24 self-dual conformal field theories is disconnected, in contrast to that of the 24 dimensional even self-dual lattices. The framework of the conformal field theories admits a richer structure than that of the lattices.

We begin by recalling the definition of neighboring lattices [2]. Two lattices A and B are said to be neighbors if their intersection has index two in each of them. We restrict attention to unimodular lattices. Choose $x \in \frac{1}{2}A - A$ such that x^2 is integral. Define $A_x = \{a \in A | a \cdot x \in \mathbb{Z}\}$. Then the lattice $B = \langle A_x, x \rangle$ is a unimodular neighbor of A . Further, all neighbors of A arise in this way. The neighborhood graph of the even self-dual lattices in 24 dimensions is then simply given by joining vertices corresponding to lattices by an edge if they are neighbors (note that in this case x^2 is even).

The notion of neighboring lattices is clearly analogous to the existence of a \mathbb{Z}_2 -orbifold construction between conformal field theories. (See [14, 19] for discussion of the analogies and deeper connections which exist between the theory of lattices and conformal field theory.) Let us make this analogy clearer. Consider two even self-dual neighboring lattices A and B (B constructed from A as above). Consider the corresponding FKS conformal field theories $\mathcal{H}(A)$ and $\mathcal{H}(B)$. Define an involution on $\mathcal{H}(A)$ by $\theta_A = e^{2\pi i x \cdot p}$, where p is the momentum operator on $\mathcal{H}(A)$. Define also an involution on $\mathcal{H}(B)$ by $\theta_B|\lambda\rangle = |\lambda\rangle$ for $\lambda \in A_x$ and $\theta_B|\lambda\rangle = -|\lambda\rangle$ for $\lambda \in x + A_x$ (and with trivial action on the bosonic creation and annihilation operators). Then the corresponding invariant sub-conformal field theories are both trivially seen to be isomorphic to $\mathcal{H}(A_x)$. In other words, the invariant Lie algebra under the automorphism θ_B on $\mathcal{H}(B)$ or θ_A on $\mathcal{H}(A)$ is that with root lattice given by the span of the length squared two vectors in A_x [14]. It is clear in this case that the corresponding orbifold theory is consistent, *i.e.* to convince oneself that the orbifold of $\mathcal{H}(A)$ with respect to θ_A is $\mathcal{H}(B)$. Hence we must have the relation (2) between the dimensions of the corresponding Lie algebras. In this case, the dimension of the Lie algebra of $\mathcal{H}(A)$ is simply $|A(2)| + 24 = 24(h(A) + 1)$, where $h(A)$ is the corresponding Coxeter number. Similarly for the theory $\mathcal{H}(B)$. The dimension of the invariant Lie subalgebra is $|A_x(2)| + 24$. Consider the odd unimodular lattice $C = A_x \cup ((A - A_x) + x)$ corresponding to the neighborhood

We may also remark that from the fact that the neighborhood graph for the Niemeier lattices is connected, that it is a subgraph of the neighborhood graph for the conformal field theories and from our results in the preceding section, we see that we may construct at most 69 and at least 66 (under the assumption that when there is a unique candidate in our test then the construction is consistent) of the 71 theories starting from any given one. In other words, perhaps all but two of the central charge 24 self-dual conformal field theories may be obtained by repeated \mathbb{Z}_2 -orbifolding of the Moonshine module using the constructions identified in Figure 1 and Table 1.

6 Extension to Third Order Twists

Let us briefly consider the extension of our method to third order twists in order to identify possible constructions of the theories not constructed in the \mathbb{Z}_2 analysis above.

Suitably rewriting the results of [20], we find that the analog of (2) is

$$c + c' = 4c_0 + 24 - 108\alpha_1 - 36\alpha_2, \quad (6)$$

where α_i is the number of states of conformal weight $i/3$ in the twisted sector (and so we assume them to be non-negative integers).

We omit the details, but we find that we obtain (as the unique candidate for a given \mathbb{Z}_3 -invariant subalgebra) $A_1^3 A_7$ from A_4^2 and $A_2 F_4$ from A_6 . However, the theory $A_1 D_5$ remains isolated.

Similar relations hold true for twists of order 5 and 7, but we do not carry through the analysis as we feel that with such high order twists on low dimensional algebras, the possibility of accidental spurious solutions is too great to ignore.

7 Conclusions

We have obtained an upper bound on the dimension of the Lie algebra of an orbifold theory corresponding to an automorphism of order two or three from modularity considerations. This serves to limit the degree of enhancement of the common \mathbb{Z}_2 -invariant algebra by the twisted sectors in the orbifold model, and so aids in many cases in the identification of a unique orbifold candidate on Schellekens' list of 71 self-dual conformal field theories at central charge 24, or alternatively the absence of a suitable candidate indicates that the given orbifold cannot be consistent.

We have thus conjectured constructions of all but one of the 71 theories. Though the definition of the automorphism on a given conformal field theory is not fully specified by its action on the corresponding Lie algebra, and in fact in many cases the automorphism does not lift to the full theory (or at least does not lift to an automorphism of the same order),

at the very least we have narrowed down the number of cases which need be considered by more explicit and tedious methods to a handful of possible constructions. The fact that the majority of the theories appear to be given simply by successive \mathbb{Z}_2 -orbifolding of a particular theory, such as the Moonshine module, should prove useful in many applications and calculations.

In addition, we have demonstrated the extension of the “neighborhood graph” of the even self-dual lattices to the self-dual conformal field theories and further demonstrated that our modular constraint may be regarded as the analog of the relation between the dimensions of the root lattices of neighboring even lattices and the corresponding odd unimodular lattice, thus extending the often surprisingly deep connections and analogies between the theory of lattices and conformal field theory suggested in [14] and continued in [8, 19, 25].

In any case, our method can be used to rather simply, using elementary Lie algebra techniques, exclude proposed orbifold constructions. In particular, it can be used as a supplement to Vafa’s “level-matching” condition, and as a replacement for the latter in cases where too little is known of the explicit structure of the conformal field theory or how to construct its orbifold. In this vein, we have demonstrated rigorously the isolation under \mathbb{Z}_2 orbifoldings of at least 2 of the 71 self-dual $c = 24$ conformal field theories, and the isolation under both \mathbb{Z}_2 and \mathbb{Z}_3 orbifoldings of the theory with algebra A_1D_5 .

The explicit construction of the self-dual theories by methods analogous to those of [6] or by the more indirect but more powerful approach of [23] will be pursued in future work.

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